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WAVE FORMATION ON THE FREE SURFACE OF OIL FILMS

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The origination of wave motion on the surface of a thin layer of oil is studied. This layer is considered as an incompressible pseudoplastic fluid. and surface tension is taken into account. It is shown analytically and numerically that these flows may be stable or unstable depending on the value of the Ostwald number. Profiles of the free surface are found for various values of the Ostwald and Weber numbers.

According to current notions, oil is an anomalously viscous non-Newtonian fluid, and its hydromechanical properties may be described using Ostwald's power-law rheological model with an exponent $n = 0.8$. It is known that wave motions of various types can be developed on the free surface of all fluids, for example, hydraulic shocks, kinematic waves, dispersion waves, etc. [1–7]. In addition, under certain parameters of the fluid and external actions, free surfaces are unstable to small and finite perturbations. As a result, these free surfaces acquire the form of randomly distributed hills and valleys, which, in turn, may distort or even interrupt the fluid flow. The study of these phenomena is very important for technological applications.

Flows of thin layers of viscous incompressible non-Newtonian fluids over inclined planes, in particular, instability of these flows and wave formation, have been extensively studied (see, for instance, [1] and the papers cited there). Undoubtedly, a detailed analysis of these phenomena requires a numerical solution of the Navier–Stokes equations in the domain with a free surface whose position changes in time. However, despite the continuously increasing performance of computers, this path is still too labor-consuming, and approximate models have been developed. For comparatively low Reynolds numbers, Benney [2] and Mei [3] derived an equation for the fluid-layer thickness and studied problems of linear stability and steady finite-amplitude waves using the method of expansion in the small parameter ε (the ratio of the fluid-layer thickness to some characteristic wavelength along the layer). In the case of small deviations of the layer thickness from the undisturbed value and high surface tension, the Kuramoto-Sivashinskii equation was derived [4], whose solutions are presented in [5, 6]. For high Reynolds numbers, under the assumption of a self-similar parabolic profile, model equations were derived for the shape of the free surface and fluid flow rate averaged over the layer thickness (see, for example, [1]). In most papers, the surface tension is assumed to be large; terms with high derivatives are retained in the equations, and the pressure is assumed to be hydrostatic. Lee and Mei [7] derived equations averaged over the layer thickness with accuracy to terms of order ε^2; such a model is valid for small and moderate Weber numbers, which characterize the ratio of the surface-tension forces to the forces of inertia or gravity.

In the present paper, we consider problems of the linear and nonlinear theory of stability of oil films (considered as a non-Newtonian fluid) flowing down an inclined plane under the action of the forces of gravity, viscosity, and surface tension; the latter is assumed to be large. The initial equations are the laws of variation of mass and momentum supplemented by Ostwald's power-law equation of state for fluids. The results of

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investigation of the linear stability of such fluids with ignored surface tension are presented in [8, 9], and a brief analysis of linear stability in the presence of surface-tension forces is given in [10].

1. Model. We consider the two-dimensional motion of a layer of an incompressible non-Newtonian fluid over a plane inclined at an angle [illegible] the horizontal plane and write the initial equations in the following form:

$$\rho(u_t + uu_x + vu_y) = -p_x + \rho g \sin\alpha + (\sigma_{xx})_x + (\tau_{xy})_y,$$
$$\rho(v_t + uv_x + vv_y) = -p_y - \rho g \cos\alpha + (\tau_{yx})_x + (\sigma_{yy})_y, \quad u_x + v_y = 0. \tag{1.1}$$

The x coordinate is directed along the inclined plane and the y coordinate is perpendicular to it, u and v are the x- and y-components of velocity, and σ and τ are the normal and tangential components of the stress tensor. For non-Newtonian fluids, they are equal to

$$\sigma_{xx} = 2\rho\nu_n A_n u_x, \quad \sigma_{yy} = 2\rho\nu_n A_n v_y, \quad \tau_{xy} = \tau_{yx} = \rho\nu_n A_n(u_y + v_x),$$
$$A_n = [2u_x^2 + 2v_y^2 + (u_y + v_x)^2]^{(n-1)/2},$$

where ρ is the density of the fluid and ν_n [$\mathrm{m}^2 \cdot \mathrm{sec}^{n-2}$] is the kinematic viscosity of the fluid with an exponent n.

Equations (1.1) should be supplemented by the boundary conditions. Adhesion on the inclined plane $y = 0$ means $u = v = 0$. On the free surface $y = H(x,t)$, the shear stress equals zero, the normal stress is compensated by surface tension, and, in addition, the standard kinematic condition is valid. Therefore, we obtain

$$p_\tau = 0, \quad p_n = -\sigma(1 + H_x^2)^{-3/2} H_{xx}, \quad H_t + uH_x = v,$$

where $p_\tau = \tau_{xy}\cos 2\theta + (1/2)(\sigma_{yy} - \sigma_{xx})\sin 2\theta$, $\tan\theta = H_x$, $p_n = -p + \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - \tau_{xy}\sin 2\theta$, and σ is a constant coefficient of surface tension. Substituting the expressions for the stress-tensor components, we write the boundary conditions in the form

$$(1 - H_x^2)(u_y + v_x) + 4H_x v_y = 0; \tag{1.2}$$

$$-p + \frac{2\rho\nu_n A_n}{1 + H_x^2}[(1 - H_x^2)v_y - H_x(u_y + v_x)] = \frac{\sigma H_{xx}}{(1 + H_x^2)^{3/2}}. \tag{1.3}$$

The velocity profile for a steady uniform flow has the form

$$u(y) = u_s\Big[1 - \Big(1 - \frac{y}{H_0}\Big)^{(n+1)/n}\Big], \quad u_s = \Big(\frac{g\sin\alpha}{\nu_n}\Big)^{1/n}\frac{n}{n+1}H^{(n+1)/n}, \tag{1.4}$$

where u_s is the fluid velocity at the free surface.

We introduce the scales L_0 and H_0 for the length along and perpendicular to the inclined plane, respectively, $p_0 = \rho u_0^2$ for pressure,

$$u_0 = \Big(\frac{g\sin\alpha}{\nu_n}\Big)^{1/n}\frac{n}{2n+1}H_0^{(n+1)/n}$$

for the longitudinal velocity, and $t_0 = L_0/u_0$ for time. L_0 is a certain characteristic wavelength of perturbations and H_0 is the undisturbed thickness of the fluid layer. We consider the long-wave approximation ($\varepsilon \equiv H_0/L_0 \ll 1$) using the above scales. As a result, the initial equations (1.1) and the boundary conditions (1.2) and (1.3) can be written in dimensionless variables with accuracy to terms of zeroth and first orders in the small parameter ε:

$$\varepsilon(u_t + uu_x + vu_y) = -\varepsilon p_x + \frac{1}{\mathrm{O}_n}\Big[\Big(\frac{2n+1}{n}\Big)^n + (A_n(u_y + \varepsilon^2 v_x))_y + 2\varepsilon^2(A_n u_x)_x\Big],$$

$$\varepsilon^2(v_t + uv_x + vv_y) = -p_y - \frac{1}{\mathrm{O}_n}\Big\{\Big(\frac{2n+1}{n}\Big)^n\cot\alpha - \varepsilon[(A_n(u_y + \varepsilon^2 v_x))_x - 2(A_n u_x)_y]\Big\},$$

$$u_x + v_y = 0.$$

Here $A_n = [(u_y + \varepsilon^2 v_x)^2 + 4\varepsilon^2 v_y^2]^{(n-1)/2}$. The boundary conditions on the bottom ($y = 0$) and on the free surface [$y = H(x,t)$] are

$$u = v = 0, \quad H_t + uH_x = v, \quad (1 - \varepsilon^2 H_x^2)(u_y + \varepsilon^2 v_x) + 4\varepsilon^2 H_x v_y = 0,$$

$$-p + \frac{2\varepsilon A_n}{(1+\varepsilon^2 H_x^2)\mathrm{O}_n}[(1-\varepsilon^2 H_x^2)v_y - H_x(u_y + \varepsilon^2 v_x)] = \frac{\varepsilon^2 \mathrm{We}_n H_{xx}}{(1+\varepsilon^2 H_x^2)^{3/2}}.$$

Here $\mathrm{O}_n = H_0^n u_0^{2-n}/\nu_n$ is the Ostwald number and $\mathrm{We}_n = \sigma H_0/(\rho Q_0^2)$ is the Weber number ($Q_0 = u_0 H_0$). We note that the small parameter ε is not independent but is determined by solving the problem posed; the explicit introduction of this parameter into derivation of the model equations allows one to separate terms of the required order.

Expanding the functions u, v, and p in powers of ε, we find the equations and boundary conditions with accuracy to first-order terms:

$$\varepsilon(u_t + uu_x + vu_y) = -\varepsilon p_x + \frac{1}{\mathrm{O}_n}\Big[\Big(\frac{2n+1}{n}\Big)^n + (|u_y|^{n-1}u_y)_y\Big],$$

$$p_y = \frac{1}{\mathrm{O}_n}\Big\{-\Big(\frac{2n+1}{n}\Big)^n \cot\alpha + \varepsilon\Big[(|u_y|^{n-1}u_y)_x - 2(|u_y|^{n-1}u_x)_y\Big]\Big\}, \quad u_x + v_y = 0; \tag{1.5}$$

$$u = v = 0 \quad \text{for} \quad y = 0,$$

$$H_t + uH_x = v, \quad u_y = 0, \quad p = -\varepsilon^2 \mathrm{We}_n H_{xx} \quad \text{for} \quad y = H(x,t). \tag{1.6}$$

To obtain the boundary condition for pressure, it is assumed that the surface tension is large: $\varepsilon^2\mathrm{We}_n \sim 1$. In Eqs. (1.5) and (1.6), the pressure is not hydrostatic, since it has a correction of order ε.

For further simplification of Eqs. (1.5), we integrate the equation for pressure with respect to y from 0 to y using the velocity profile (1.4) with u_s and H depending on x and t, as is done in Kármán's integral method. As a result, we obtain

$$p = \frac{1}{\mathrm{O}_n}\Big(\frac{2n+1}{n}\Big)^n\{(H-y)\cot\alpha + \varepsilon H_x(H-y)[2H^{1/n}(H-y)^{-1/n} - 1]\} - \varepsilon^2\mathrm{We}_n H_{xxx}. \tag{1.7}$$

Differentiation of (1.7) with respect to x yields

$$p_x = \frac{1}{\mathrm{O}_n}\Big(\frac{2n+1}{n}\Big)^n\Big\{H_x\cot\alpha - \varepsilon\Big[H_x^2\Big(2H^{1/n}(H-y)^{-1/n}\Big(1 - \frac{y}{nH}\Big) - 1\Big)$$

$$+ (H-y)H_{xx}(2H^{1/n}(H-y)^{-1/n} - 1)\Big]\Big\} - \varepsilon^2\mathrm{We}_n H_{xxx}. \tag{1.8}$$

After that, we integrate (1.5) over the layer thickness using the boundary conditions (1.6) and the approximate velocity profile (1.4) to relate the mean quantities $\langle u\rangle$ and $\langle u^2\rangle$, namely, $\langle u^2\rangle = (4n+2)\langle u\rangle^2/(3n+2)$, and also expression (1.8) for the longitudinal pressure gradient. As a result, we obtain

$$H_t + Q_x = 0, \tag{1.9}$$

$$Q_t + \frac{4n+2}{3n+2}\Big(\frac{Q^2}{H}\Big)_x = \frac{1}{\varepsilon\mathrm{O}_n}\Big(\frac{2n+1}{n}\Big)^n\Big[(1 - \varepsilon H_x\cot\alpha)H - \frac{Q^n}{H^{2n}}\Big] + \varepsilon^2\mathrm{We}_n H H_{xxx}.$$

The above correction to the hydrostatic value of pressure is proportional to ε [see (1.7)]. Since we consider equations valid with accuracy to first-order terms in the small parameter, there is no correction in Eqs. (1.5), since the pressure gradient in the x direction enters into the equations with a factor ε. This means that the pressure is hydrostatic in equations obtained for the free-surface shape and fluid flow rate. We now pass to studying the properties of Eqs. (1.9) where we assume that $\varepsilon = 1$. This does not lead to loss of generality, since the transition to dimensional variables reduces only to changing the scales. A comparison of the rejected and retained terms in (1.9) based on results of the numerical solution shows that the former are no more than 0.1 of the latter.

2. Linear Analysis. Similarly to [7–9], we linearize system (1.9) relative to small perturbations of a moving uniform layer of constant thickness assuming that $H = 1 + h$ and $Q = 1 + q$ $(h, q \ll 1)$. Representing the solution of the thus-obtained system of two linear equations in the form of periodic waves $h, q \sim \exp(i(kx - \omega t))$, we obtain

$$\gamma = b(v_0 - v)/(2(v - a)); \tag{2.1}$$

$$\mathrm{We}_n k^4 - (v^2 - 2av + c)k^2 - \frac{b^2(v - v_0)(v + v_0 - 2a)}{4(v - a)^2} = 0. \tag{2.2}$$

Here ω is the complex frequency, k is the real wavenumber of small perturbations, γ is the growth rate, v is the phase velocity of small perturbations, $a = (4n+2)/(3n+2) = 1.18$, $b = ((2n+1)/n)^n(n/\mathrm{O}_n) = 2.05/\mathrm{O}_n$, $c = a - (b/n)\cot\alpha = 1.18 - (2.05\cot\alpha)/\mathrm{O}_n$, and $v_0 = (2n+1)/n = 3.25$.

With accuracy to notation, Eqs. (2.1) and (2.2) coincide with the corresponding equations in the cases $\mathrm{We}_n = 0$ [8, 9] and $\mathrm{We}_n \neq 0$ [10]. We also note that, in the particular case of a Newtonian liquid ($n = 1$), Eqs. (2.1) and (2.2) are transformed to those given in [11, 12].

It follows from (2.1) that the growth rate γ is positive for $v < v_0$ and negative for $v > v_0$. Thus, the examined motion of a uniform layer of oil is unstable to periodic small perturbations with phase velocities $v < 3.25$ and stable to small perturbations with phase velocities $v > 3.25$. For comparison, we note that we have $v_0 = 3$ for a Newtonian fluid.

From Eq. (2.2) we have

$$k^2 = \frac{(v - v_1)(v - v_2)}{2\mathrm{We}_n}\Big\{1 \pm \Big[1 - \frac{\mathrm{We}_n b^2(v_0 - v)(v_0 + v - 2a)}{(v - a)^2(v - v_1)^2(v - v_2)^2}\Big]^{1/2}\Big\},$$

where $v_1 = a + (a^2 - c)^{1/2}$ and $v_2 = a - (a^2 - c)^{1/2}$. The growth rate γ turns into zero if $v = v_0$, i.e., for two values of the wavenumber

$$k^2 = 0, \qquad k^2 \equiv k_*^2 = (v_0 - v_1)(v_0 - v_2)/\mathrm{We}_n.$$

Substituting the values of v_0, v_1, and v_2, we find the square of the boundary wavenumber:

$$k_*^2 = \frac{2n + 1}{\mathrm{We}_n n^2}\Big[1 - (2n + 1)^{n-1} n^{2-n}\frac{\cot\alpha}{\mathrm{O}_n}\Big]. \tag{2.3}$$

It follows from this formula that the examined flow is stable ($\gamma \leqslant 0$) for $\mathrm{O}_n \leqslant \mathrm{O}_n^*$, where $\mathrm{O}_n^* = (2n + 1)^{n-1} n^{2-n}\cot\alpha = 0.63\cot\alpha$ is the critical Ostwald number. For the case of fluid motion over a vertical wall ($\alpha = 90°$), the critical Ostwald number is equal to zero, and the perturbations are unstable for all Ostwald numbers. If the Ostwald number is greater than the critical value, there exists a finite range of wavenumbers $\Delta k = 0$–k_* in which small perturbations are unstable (the growth rate is positive). If surface tension is ignored, the region of unstable wavenumbers is unbounded ($k \to \infty$) [9]. Surface tension stabilized small-scale perturbations, making the region of unstable wavenumbers finite. From formula (2.3) we derive the equation of the neutral curve $\mathrm{O}_n = \mathrm{O}_n(k)$, which separates the regions of stability and instability:

$$\mathrm{O}_n = \mathrm{O}_n^*/(1 - \mathrm{We}_n n^2 k^2/(2n + 1)).$$

Since oil has different properties in different deposits, the mean parameters were used in calculations: $\rho = 850$ kg/m^3, $\nu_n = 0.001$ m$^2\cdot$sec$^{-1.2}$, and $\sigma = 0.026$ N/m. In all the calculations, we had $\alpha = 45°$. Figure 1 shows the neutral curves for oil with the above-mentioned parameters for different Weber numbers. The regions above and to the left of the neutral curve correspond to stability and those below and to the right of the neutral curve to instability. Figure 2 shows the growth rate of small perturbations γ as a function of the wavenumber k for oil with the parameters $\mathrm{O}_n = 1$, $\mathrm{We}_n = 108$, and $n = 0.8$, which corresponds to a layer thickness $H_0 = 0.47$ cm. Taking into account that the time scale t_0 [sec] is

$$t_0 = \Big(\frac{H_0}{g\sin\alpha}\Big)^{1/2}\Big(\frac{2n + 1}{n}\Big)^{n/2}\frac{1}{\mathrm{O}_n^{1/2}},$$

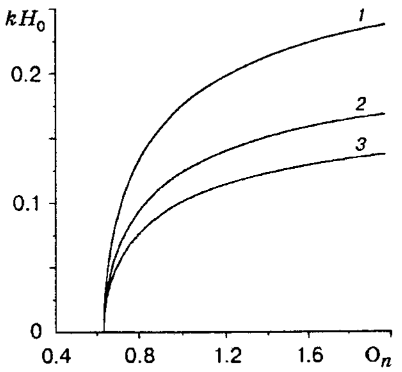

Fig. 1

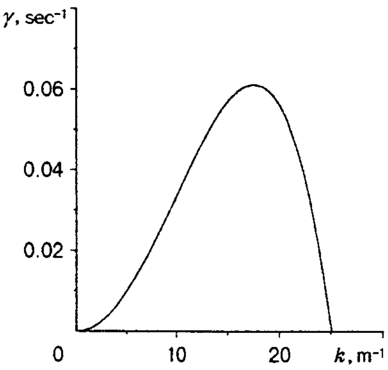

Fig. 2

Fig. 1. Neutral stability curves for oil ($n = 0.8$ and $\mathrm{O}_n^* = 0.63$): $\mathrm{We}_n = 50$ (1), 100 (2), and 150 (3).

Fig. 2. Growth rate versus the wavenumber for a flow with $\mathrm{O}_n = 1$, $\mathrm{We}_n = 108$, and $n = 0.8$.

we can write the dimensional values of the growth rate γ [sec^{-1}] and phase velocity $\bar{v}$ [m/sec]:

$$\bar{\gamma} = \Big(\frac{g\sin\alpha}{H_0}\Big)^{1/2}\Big(\frac{n}{2n+1}\Big)^{n/2}\mathrm{O}_n^{1/2}\gamma, \qquad \bar{v} = (gH_0\sin\alpha)^{1/2}\Big(\frac{n}{2n+1}\Big)^{n/2}\mathrm{O}_n^{1/2}v.$$

The dimensional $\bar{k}$ [m^{-1}] and dimensionless k wavenumbers are related as $\bar{k} = k/H_0$. For oil with the above-mentioned parameters, we have $u_0 = 11.3$ cm/sec, $v_0 = 36.8$ cm/sec, $v_0/u_0 = 3.25$, and $\gamma_{\max} = 0.06$ sec^{-1} for $\bar{k}_{\max} = 17.7$ m^{-1}, the boundary wavenumber is $k_* = 25$ m^{-1}, and perturbations with wavelengths from $\lambda_{\max} = \infty$ to $\lambda_{\min} = 25$ cm are unstable. All formulas of this section are transformed to the corresponding formulas for a viscous Newtonian fluid [11]. Therefore, the numerical values of the characteristic quantities given here are adequate to the same extent as the flow characteristics in the case of a Newtonian fluid, and the latter are in good agreement with the experimental results of [11].

3. Weak Nonlinearity. We consider the case of weak nonlinearity adding, as previously, small perturbations $H = 1 + h$ and $Q = 1 + q$ ($h, q \ll 1$) to a uniform flow but take into account terms up to the second order in ε. In this approximation, Eqs. (1.9) may be written in the following form:

$$h_t + q_x = 0; \tag{3.1}$$

$$q_t + a(2q - h)_x - (b/n)[(2n+1)h - h_x\cot\alpha - nq] - \mathrm{We}_n h_{xxx}$$

$$= 2a(q - h)(h - q)_x + (b/n)[2n^2qh - hh_x\cot\alpha - 0.5n(n-1)q^2 - n(2n+1)h^2] + \mathrm{We}_n hh_{xxx}. \tag{3.2}$$

Differentiating (3.2) with respect to x and (3.1) with respect to t and eliminating the derivatives of q, we obtain

$$h_t + v_0h_x + b^{-1}(h_{tt} + 2ah_{xt} + ch_{xx} + \mathrm{We}_n h_{xxxx}) = (2a/b)[(qq_x)_x + (hh_x)_x - (qh)_{xx}]$$

$$- 2n(qh)_x + (1/n)(hh_x)_x\cot\alpha + (n-1)qq_x + 2(2n+1)hh_x - (\mathrm{We}_n/b)(hh_{xxx})_x.$$

In accordance with [11–13], we can assume, without loss of accuracy, that $q = v_0h$ and $\partial/\partial t = -v_0\partial/\partial x$ in nonlinear terms, eliminate the function q, and obtain an evolution equation for the free-surface shape:

$$h_t + v_0h_x + v_0\frac{n+1}{n}hh_x + \frac{\mathrm{O}_n}{nv_0^n}\Big(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + v_2\frac{\partial}{\partial x}\Big)h$$

$$+ \frac{\mathrm{We}_n\mathrm{O}_n}{nv_0^n}[(1+h)h_{xxx}]_x = \frac{\mathrm{O}_n}{n}\Big[\frac{4(2n+1)}{(3n+2)v_0^n}\Big(\frac{n+1}{n}\Big)^2 + \frac{\cot\alpha}{\mathrm{O}_n}\Big](hh_x)_x. \tag{3.3}$$

The first two terms in the left side of (3.3) describe a kinematic wave propagating along the inclined plane with a velocity $v_0 = 3.25$, the third term corresponds to quadratic nonlinearity, the fourth term describes an inertial wave of higher order as compared to the kinematic wave, and the last term appears owing to surface tension. The right side of the equation corresponds to nonlinear diffusion.

We consider two limiting cases: for moderate $O_n \approx 1$ and high Ostwald numbers $O_n \gg 1$. In the limiting case $O_n \approx 1$, the terms that describe the kinematic wave play the governing role; therefore, in the terms corresponding to the inertial wave, we can make the substitution $\partial/\partial t = -v_0 \partial/\partial x$ and ignore the right side, which leads to the equation

$$h_t + v_0 h_x + v_0 \frac{n+1}{n} h h_x + \frac{O_n}{n^2 v_0^{n-1}}\Big(1 - \frac{O_n^*}{O_n}\Big) h_{xx} + \frac{\mathrm{We}_n O_n}{n v_0^n} h_{xxxx} = 0. \tag{3.4}$$

The sign of the term corresponding to diffusion depends on the Ostwald number. If $O_n < O_n^*$, this term is negative and describes the usual diffusion, which tends to smooth the perturbations of the free surface of the oil film. If $O_n > O_n^*$, this term is positive and describes the growth in the amplitude of perturbations, which is related to the "negative viscosity." In this case, the energy from the mean flow is transferred to the kinematic wave through the inertial wave.

In the limiting case $O_n \gg 1$, we introduce a new small parameter $\varepsilon_1 = n v_0^n / O_n$ and write Eq. (3.3) in the following form:

$$\Big(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial x}\Big) h - \Big[\frac{8n+4}{3n+2}\Big(\frac{n+1}{n}\Big)^2 + \frac{v_0^n \cot\alpha}{O_n}\Big](h h_x)_x$$

$$+ \mathrm{We}_n[(1+h)h_{xxx}]_x + \varepsilon_1 \Big[h_t + v_0\Big(1 + \frac{n+1}{n} h\Big) h_x\Big] = 0. \tag{3.5}$$

It follows from this equation that the governing role belongs to inertial waves propagating along the flow with a velocity v_1 and opposite the flow with a velocity v_2. Separating the streamwise wave, we make the substitution $\partial/\partial t = -v_1 \partial/\partial x$ in terms containing time derivatives. As a result, we obtain

$$(v_2 - v_1)(h_t + v_1 h_x)_x - \frac{8n+4}{3n+2}\Big(\frac{n+1}{n}\Big)^2 (h h_x)_x$$

$$+ \mathrm{We}_n[(1+h)h_{xxx}]_x + \varepsilon_1 \Big[(v_0 - v_1)h + v_0 \frac{n+1}{2n} h^2\Big]_x = 0. \tag{3.6}$$

Note that we ignored the term $v_0^n \cot\alpha / O_n$ in deriving this equation, since $O_n \gg 1$. Integration with respect to x yields

$$h_t + v_1 h_x + \frac{1}{v_1 - v_2}\Big[\frac{8n+4}{3n+2}\Big(\frac{n+1}{n}\Big)^2 h h_x - \mathrm{We}_n (1+h) h_{xxx} - \varepsilon_1 \Big(v_0 - v_1 + v_0 \frac{n+1}{n} h\Big) h\Big] = 0, \tag{3.7}$$

where, in the same approximation ($O_n \gg 1$), we obtain

$$v_1 = 2(2n+1)\Big(1 + \sqrt{\frac{n}{4n+2}}\Big)\Big/(3n+2) = 1.64,$$

$$v_2 = 2(2n+1)\Big(1 - \sqrt{\frac{n}{4n+2}}\Big)\Big/(3n+2) = 0.72, \quad v_1 - v_2 = 2\sqrt{2n(2n+1)}/(3n+2) = 0.92.$$

For a Newtonian fluid ($n = 1$) and $\alpha = 90°$, this equation was derived in [12]. It follows from (3.6) that, for high O_n, the kinematic wave pumps the energy to the inertial wave; this process of "low-frequency pumping" is described by a linear term [11]. Linearization of Eq. (3.7) with neglect of the small pumping of energy yields the dispersion equation

$$\frac{\omega_r}{k} = v_1\Big(1 + \frac{\mathrm{We}_n k^2}{(v_1 - v_2) v_1}\Big),$$

from which it follows that the phase velocity of inertial waves depends on the wavenumber, increasing with the growth in the latter. Hence, inertial waves have a positive dispersion, since the coefficient at We_n is positive (see, e.g., [14]). This dispersion caused by surface tension leads to the appearance of "ripples" ahead of the inertial wave. The dispersion length squared is

$$l^2 = \frac{\mathrm{We}_n(3n+2)^2}{4n(2n+1)(1+2\sqrt{1+1/(2n)})} = 0.66\,\mathrm{We}_n.$$

If the pumping of energy is ignored, Eq. (3.7) in a coordinate system moving with a velocity v_1 acquires the form of a quasi-linear transport equation with the dispersion

$$h_t + 2\sqrt{1+\frac{1}{2n}}\Big(\frac{n+1}{n}\Big)^2 hh_x - \frac{\mathrm{We}_n(3n+2)}{2\sqrt{2n(2n+1)}}\,h_{xxx} = 0.$$

Using the substitution

$$h = \bar{h}\Big(\frac{n}{n+1}\Big)^2\Big(\frac{n}{4n+2}\Big)^{1/2},$$

we obtain the Korteweg–de Vries equation

$$\bar{h}_t + \bar{h}\,\bar{h}_x + \beta\bar{h}_{xxx} = 0,$$

where $\beta = -\mathrm{We}_n(3n+2)/(2\sqrt{2n(2n+1)}) = 1.08\,\mathrm{We}_n$, whose solutions are known.

Equations (3.1)–(3.7) derived in the approximation of weak nonlinearity allow one, in the limiting cases of low and high Ostwald numbers, to describe the wave processes considered and obtain quantitative estimates of their parameters. For arbitrary Ostwald numbers, these equations can be solved only numerically. If we study the evolution of perturbations not assuming the nonlinearity to be small, we have to consider Eqs. (1.9).

4. Numerical Solution of Nonlinear Equations. The evolution of finite-amplitude perturbations was studied numerically. To solve system (1.9) with ignored forces of surface tension, Berezin et al. [9] used an explicit finite-difference scheme in which the fluxes of mass and momentum were approximated by one-sided differences in accordance with the flow direction, and the term HH_x proportional to the pressure gradient was approximated by the central difference. This scheme possesses a conventional stability; the ratio of steps $\delta t/\delta x$ necessary for stability was chosen using auxiliary calculations. Taking into account surface tension increases the order of the highest derivative with respect to the coordinate up to the third one. This scheme supplemented by a symmetric finite difference for approximation of the third derivative was used to solve system (1.9):

$$H1_i = H_i - (\delta t/\delta x)(u_{i+0.5}H_i - u_{i-0.5}H_{i-1}),$$

$$Q1_i = Q_i - a_n(\delta t/\delta x)(u_{i+0.5}u_iH_i - u_{i-0.5}u_{i-1}H_{i-1}) - b_n\cot\alpha(\delta t/(2\delta x))H_i(H_{i+1} - H_{i-1})$$

$$+ b_n\delta t[H_i - Q_i^n/H_i^{2n}] + \mathrm{We}_n(\delta t/(2\delta x^3))H_i(H_{i+2} - 2H_{i+1} + 2H_{i-1} - H_{i-2}).$$

Here $H1_i \equiv H_i^{m+1}$, $H_i \equiv H_i^m$, $Q1_i \equiv Q_i^{m+1}$, $Q_i \equiv Q_i^m$, $t^{m+1} = (m+1)\delta t$, $t^m = m\delta t$, $u_{i+0.5} = (u_i + u_{i+1})/2$, $u_{i-0.5} = (u_i + u_{i-1})/2$, $a_n = (4n+2)/(3n+2)$, and $b_n = \mathrm{O}_n^{-1}((2n+1)/n)^n$. The scheme is convenient in implementation, and its conventional stability, which requires choosing $\delta t < \delta x^3$, is not a great constraint because of the one-dimensionality of the problem. All the calculations were performed for the case $\alpha = 45°$.

An initial localized perturbation was chosen in the form of a smoothed step unbounded upstream, and the boundary conditions were $H(0,t) = 1 + H_1$ and $H(x_{\max}, t) = 1$, where H_1 is the amplitude of the step. The use of these boundary conditions implies that the numerical solution is continued as long as the perturbation is rather far from the computational-domain boundaries (this condition is implied in the computational algorithm).

Figure 3a shows the profile of the free surface of an oil film moving down an inclined plane at times $t = 50t_0$ and $100t_0$ after a sudden liberation of the initial step whose height is $H_1 = 0.047$ cm for $\mathrm{O}_n = 1$ and $\mathrm{We}_n = 108$, and $n = 0.8$. Since the Ostwald number is higher than the critical value 0.63, the initial perturbation is unstable, and a wave with a steep front and noticeable spatial oscillations before and behind the front appears with time. The amplitude of oscillations does not increase with time, which indicates the compensation for nonlinear effects leading to an increase in perturbation amplitudes by the stabilizing action of the surface-tension forces. If the thickness of the fluid layer increases, this leads to an increase in the Ostwald number and to a decrease in the Weber number. Figure 3b shows the profiles of the free surface of

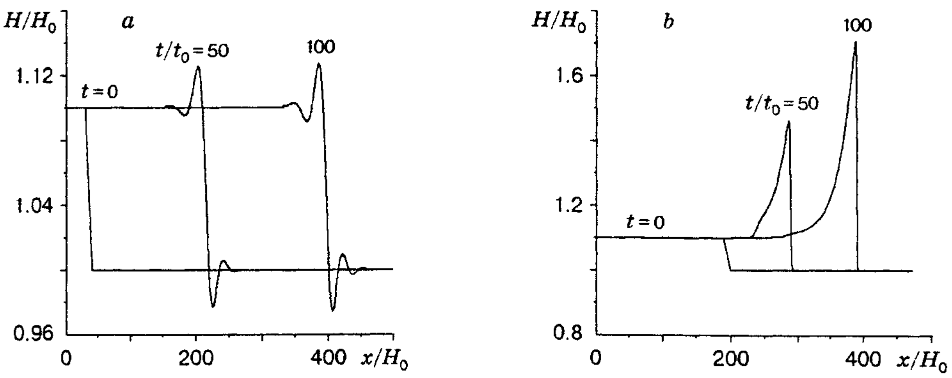

Fig. 3. Profile of the free surface of an oil layer (n = 0.8): (a) O_n = 1 and We_n = 108; (b) O_n = 160 and We_n = 0.0088.

an oil layer, which has an undisturbed thickness H_0 = 2 cm, at times $t = 50t_0$ and $100t_0$. In this case, we have O_n = 160 and We_n = 0.0088, and n = 0.8; the influence of surface tension is insignificant. The initial step changes its shape with time, and a structure of the shock-wave type arises, which resembles the steady solutions of equations averaged over the thickness of the fluid layer, which were analyzed in [8]. The profile of the free surface has a smooth sector whose thickness increases monotonically toward the front, acquires a maximum value of 3.4 cm ($t = 100t_0$), and then dramatically decreases, approaching monotonically the free-stream value.

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